



Ollscoil Chathair
Bhaile Átha Cliath
Dublin City University

ZADIE

Functional Thinking through Patterning

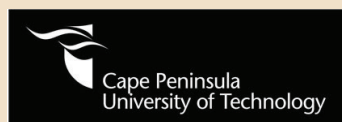
Teacher Manual

Aisling Twohill
Dublin City University, Ireland

Sharon Mc Auliffe
Cape Peninsula University of Technology, South Africa

Anna Susanne Steinweg
University of Bamberg, Germany

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ZADIE Functional Thinking through Patterning: Teacher Manual

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ZADIE Project Team

- Dr Aisling Twohill, Dublin City University, Ireland,
<https://orcid.org/0000-0003-1156-9983>
- Prof Anna Steinweg, University of Bamberg, Germany,
<https://orcid.org/0000-0002-9093-547X>
- Dr Sharon McAuliffe, Cape Peninsula University of Technology,
<https://orcid.org/0000-0003-2777-9504>



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1

Overview

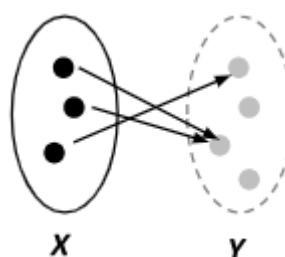
Overview

This teacher manual presents a unit of work that comprises four lessons exploring pattern, structure and algebraic thinking, developed as part of the ZADIE project. The lessons focus on functional thinking as one of the main strands of algebraic thinking (Kaput, 2008). Lesson worksheets are included as an appendix, and the starting point of each lesson is a geometrical pattern (of squares or sticks) to be explored by the children. The four lessons address the ‘big ideas’ of functional thinking through patterning activities.

Functions

“Any expression made up of a variable and some constants” (Johann Bernoulli, 1667-1748).

“A function may be thought of a rule, or correspondence, that associates with element of a set X one and only one element of a set Y ” (Swokowski, 1979).



Functional Thinking

“Functional thinking relates to understanding the notion of change and how varying quantities (or variables) relate to one another” (Wilkie, 2015).

Patterns and Functional Thinking

Figural and Numerical Perspectives

When exploring a geometrical pattern of squares or sticks, learners are expected to observe both a *spatial* structure and a *numerical* structure and to consider the relationship between the two (Radford, 2011). In this unit of work, we interchange the terms spatial and figural as both refer to the presentation of the squares or sticks of the pattern. Children will use figural and numerical approaches and, where relevant, we highlight examples of both. Some children will need to be encouraged to consider the figural aspects if they are overly numerical in their descriptions of the patterns, and vice versa.

The position number

The quantity of squares, dots or sticks needs to be expressed in relation to the *position* of the figure in the pattern (e.g., 1st, 2nd, 3rd). This ordinal *number* is represented by number cards in the ZADIE patterns (e.g. 1, 2, 3, for the 1st, 2nd, 3rd terms in Figures 1 to 4 below).

The use of number cards encourages pupils to view this number as “a sign pointing to the position” of each figure in the pattern (Warren & Cooper, 2008, p. 178).

To recognise and understand the linkage between the position of a figure in the pattern and the structure of the figure (amount, shape) is important for children to be able to describe the pattern in general, and to think about far-off terms.

Three perspectives on patterns

Three perspectives on patterns are crucial for functional thinking:

- recursive patterning
- co-variational thinking
- correspondence relationship (explicit rule) (Smith, 2008).

While working through the lessons, children will demonstrate thinking that is indicative of these perspectives. At times, they will be appropriate, but children may also need to be encouraged to broaden their perspective.

This is particularly true where children focus on recursive thinking and are therefore struggling to identify a far term.

Recursive Patterning

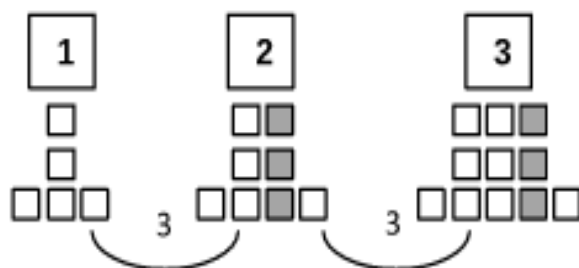


Figure 1: a recursive approach

Exploring and finding variation within a pattern by comparing the quantity of components from one figure to the next is called a *recursive* approach. For example, in Figure 1, there is a difference of 3 in the quantity of squares in each subsequent figure in the pattern.

Co-variational Thinking

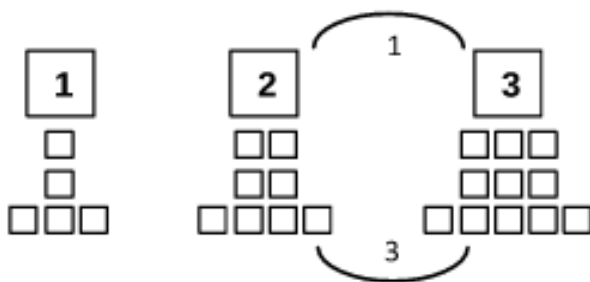


Figure 2: co-variational thinking

Analysing how two quantities, i.e., the position (ordinal) and the quantity (cardinal), vary simultaneously is called *co-variational thinking*. In Figure 2, the increase of the position (independent variable) by one causes an increase by three in the quantity (dependent variable). The co-variation describes the change that works for each and every relationship between the independent and dependent variable in the specific pattern.

The Correspondence Relationship, or Explicit Rule

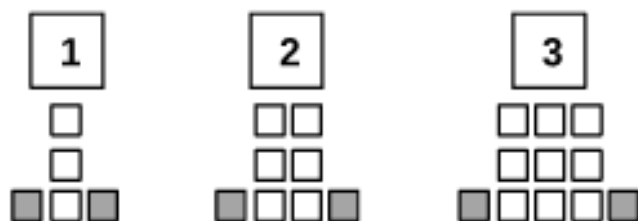


Figure 3: distinguishing constant elements from elements that are changing

Identifying the change needs to go hand in hand with identifying *constant elements*. Geometrical patterns provide a spatial structure to help in identifying the constant part and marking it (Figure 3).

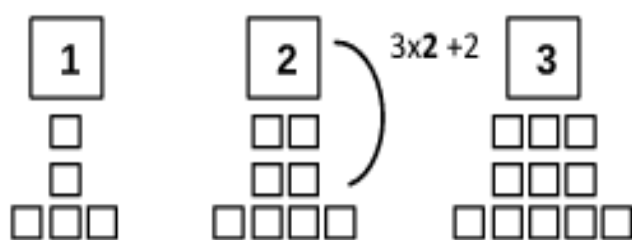


Figure 4: identifying an explicit rule

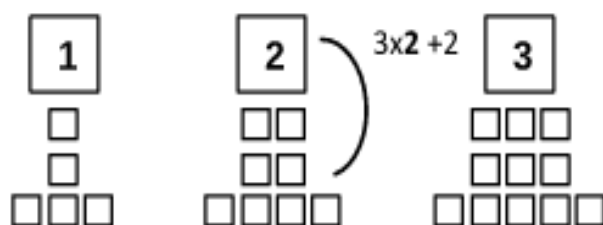
The *correspondence relationship* identifies a correlation between the independent (ordinal) and dependent (cardinal) variables. This is also referred to as an explicit rule. In Figure 4 the relationship for the 2nd figure ($n=2$) is identified as $3 \times 2 + 2$. The explicit rule for each figure in this pattern is $3n+2$.

The explicit formula represents “a functional relationship between position and numerical value of an element” (Strømskag, 2015, p. 475). The relationship between the position number, the variables and the constant elements needs to be condensed in one expression.

Moving between representations

Understanding of a mathematical idea is often associated with the flexible use of different representations. The abstract idea of a function needs to be translated into representations in classroom mathematics. Diagrams, graphs, and tables accompany descriptions of function rules in formal expressions in secondary maths education. Functional thinking involves moving flexibly between these representations.

The patterns used in this unit are one possible representation of a relationship between two sets of natural numbers. Consequently, working on the pattern of squares and (match) sticks is working on functional thinking. The potential to shift from one representation (spatial pattern) to another (numerical expressions) is central to patterning activities.



In describing figure 2 as $3 \times 2 + 2$, children are moving between the spatial and numerical representations.




The specific activities of *recognise - continue - describe - explain regularity* in elementary patterning (Steinweg et al., 2018) are the pathway to generalising and the understanding of functional relations.

The use of tables and graphs is not mandatory to allow understanding. In fact, there are concerns about using tables because the transfer of “learners’ attention from the original problem situation to a table of values means that they can lose contact with three important elements to assist with generality” (Hewitt, 2019). The children may lose contact with the structural properties, and may be distracted from explicit rules to focus on recursive thinking. Also, the validation of a rule found for the values in the table may hinder justification which depends on the structural aspects of the pattern.

Tables should, therefore, include the notation of the ‘arithmetic they would do’ (Hewitt, 2019), without calculating the quantities, as demonstrated here (see overleaf).

This teaching idea is in line with other research findings which indicate the figural reasoning approach as “more effective than the numerical reasoning approach in generalizing patterns since it involves recognizing relationships between parts of the figures forming the pattern and the step number” (El Mouhayar & Jurdak, 2016, p. 212).

Table 1: Examples of possible notations for the pattern in Figures 1-4.

1	2	3	4	...	947	...	n
							
$1+3 \times 1+1$	$1+3 \times 2+1$	$1+3 \times 3+1$	$1+3 \times 4+1$		$1+3 \times 947+1$		$1+3 \times n+1$
$3 \times 1 + 2$	$3 \times 2 + 2$	$3 \times 3 + 2$	$3 \times 4 + 2$		$3 \times 947 + 2$		$3 \times n + 2$
3 times one and two more	3 times two and two more	3 times three and two more					3 times the position number and two more

Primary school lessons do not necessarily expand the table up to the n-column but focus on far figures (like the 947th) to generalize the pattern with quasi-variables (947 as a random number, regarded as any number).

Growth points of functional thinking and the role of language

Functional thinking as a complex construct develops over time. This unit on patterns aims to support this development. It may be useful to consider this as one element of a longer developmental pathway in functional thinking. In theory, informal approaches typically build towards formal and abstract approaches. Roughly sketched, important growth points can be identified in children's abilities in working on a pattern and the different characteristic of descriptions offered:

- recognise commonality or regularity and continue the pattern (next figure), verbally describe some aspects of the pattern
- find near and next figures with reasoning, verbally describe the pattern
- find near and far figures with reasoning, verbally describe the pattern explicitly
- generate far figures by using the rule, describe the rule in symbolic notation (Twohill, 2018, p. 62).

These different steps may be useful in categorising children's responses. It is important to consider children's ideas in classroom discussions and to encourage children forward from their current developmental level.

Language is an important tool in these generalising activities. With reference to Mason (2008) every utterance must be appreciated and the "learners need to be encouraged and trusted to develop their own thinking and understanding through pattern activities and experiences which are structured to encourage learners to make generalisations, to explain and justify their generalisations and to convince one another" (Mc Auliffe, 2013, p. 60). It is, therefore, important to encourage the children to express their findings in words orally or in a written form. Teachers must keep in mind that the written forms sometime do not reflect the full scope of the ideas. The oral classroom discussion may enable the children to express themselves in more detail and to pick up ideas other children came up with. Teachers should use accurate mathematical terms as a model for children.

Another essential aspect in the realm of functional thinking and language is the justification of ideas. Formal mathematical proofs are neither expected nor necessary in primary school. However, it is crucial to motivate children not only to come up with a solution but also to describe their reasoning. Activating questions to justify the findings can be 'How did you make it?', 'How did you know?', 'Can you convince others in the panel/in the group?' etc.

Learning Activities – Structure of the Unit

The unit of work offers teaching and learning activities designed to facilitate aspects of functional thinking, and follows the growth points of the development of functional thinking outlined above.

Big ideas in this development are identified and grouped in four lessons, as followed:

- Lesson 1: Recognising geometrical patterns as sequences of ordered growing figures with a certain position number each.
 Finding next figures in given patterns.
- Lesson 2: Finding near figures in the pattern (specialising).
 Finding far figures in the pattern (generalising).
- Lesson 3: Identifying the two important elements (constant and change) of an explicit expression by colouring (a) the constant element and (b) identifying the position number in each figure.
- Lesson 4: Finding an explicit rule.
 Justifying the rule, e.g., by colouring.
 Comparing patterns (with the same rate of change) and their rules.

For each of the four lessons two worksheets are provided, with patterns that have been selected to support these big ideas.

Teaching Approaches

The activities in this unit are quite challenging and require problem-solving strategies. Each child should be given time to come up with her or his individual thoughts and comments.

Usually, in a problem-solving classroom, the teacher reads aloud the worksheet questions, and the children work in pairs or small groups on the pattern. In these ZADIE lessons we suggest that learning activities are preceded by individual working time before the individual ideas are shared in pairs, small groups or plenary (inspired by e.g. Gallin, 2010). Research results on problem solving of challenging tasks proved the importance of individual thoughts: If a solving process starts in small group discussions directly, individual ideas cannot arise by all children in the group. The problem can be tackled more effectively, if all ideas have their share in a co-constructive interaction (Götze, 2007).

Within this think & share approach the teacher acts as a facilitator and poses a question or a problem to the children as usual. Afterwards the children are given sufficient time (about 5 min) to think and gather their very own thoughts individually before sharing these ideas with others in pairs, groups, or in plenary. The individual working time allows each child to think and to practise writing down their own ideas.

The description the children give may not reflect their individual thinking to its full extent. In other words, giving an oral or written description is challenging for the children. The worksheets are personal documents, on which children practise phrasing and wording individually. Language is crucial in developing functional thinking, but teachers do not mark or correct these exercises. However, in the plenary discussion at the end of each lesson, the teacher stresses possible descriptions, uses mathematical terms, and supports the children who struggle to participate.

The discussion of the findings and approaches acknowledges and values the original ideas of the children. The teacher moderates the discussion and co-ordinates collecting various ideas on the (white) board. In this phase, it might be necessary to blend in not yet mentioned ideas, which are necessary for developing functional thinking. These other ideas may include, if needed, mathematically sound and common solutions of the problem(s) discussed and worked on. The blended in ideas should be framed as answers from another classroom. Links between the children's ideas in the actual class and these other (blended in) ideas are easier picked up, if the new idea does not stand out by being the teachers' idea.



2

The Unit of Work: Four Lessons

The Unit of Work: Four Lessons

Lesson 1

Two worksheets “Each figure has a number.” and “Next please!”

Big Ideas

1. Recognising geometrical patterns as sequences of ordered growing figures with a certain position number each.
2. Finding next figures in given patterns.

Because this lesson is the first in the unit, it is important to provide plenty of time to appreciate the individual ideas emerging from working on the patterns. From this first lesson onwards, the children should be encouraged to write down their own ideas and findings (in words).

Each figure has a number

To generalise a rule, first of all, it is important to recognise and understand the link between the *position* of a figure in the pattern and the *structure* of the figure (quantity of components, shape). This first activity stresses the idea of the position number, represented by index cards / number cards.

The two activities in “Each figure has a number” ask for any kind of relationship between the numbers on the cards and the figures (task A) and for matching the fitting number to a figure in the pattern (task B). The children’s spontaneous reactions to these questions provide helpful insight into their individual understandings.

Both activities focus the attention on the ordinal position of the figures and the ordering (growing). However, the answers to the question, how the number cards fit to the figures, cannot be given without referring back to the structure of the figure.

Possible reactions focussing on figural aspects

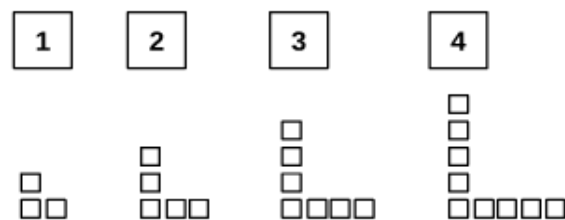
- it is getting bigger and the numbers are getting bigger
- the shape of the figure is recognised as growing

Possible reactions focussing on numerical aspects

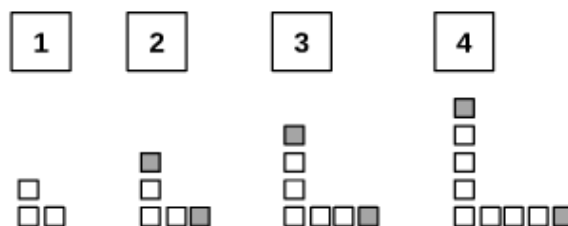
- there are more squares and the numbers get higher (cardinality)
- the change (growth) of the quantity of squares from one figure to the next one is noticed

Possible reactions recognising the ordinal number as one quantity in the pattern

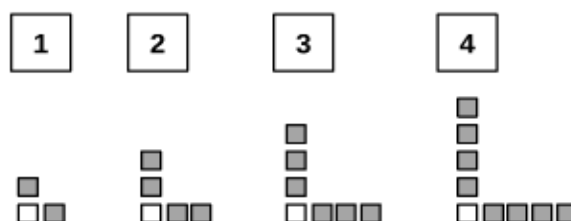
- the number is the length of one arm of the L (task A)
- one side of the rectangle is the number (task B)



The quantity of the squares 3, 5, 7, 9 ... is identified as growing



The change (growth) from figure to figure is highlighted in the pattern



The position number is identified in the pattern (twice in each figure in this example).



Growth is indicated by one more in each row and column



Growth is indicated by highlighting all additional squares

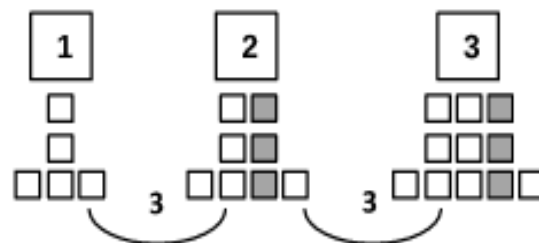
Within the discussion, the teacher may stress mathematically sound ideas or blend in possible but missing findings. It is not intended to find an explicit rule in this first lesson. The pattern given in worksheet 1, task A, is addressed again in lesson four, where the explicit rule is requested. The big idea in this first encounter, which the children have to recognise, is that the quantity of squares in the figure is not the (position) number of the figure.

Next please!

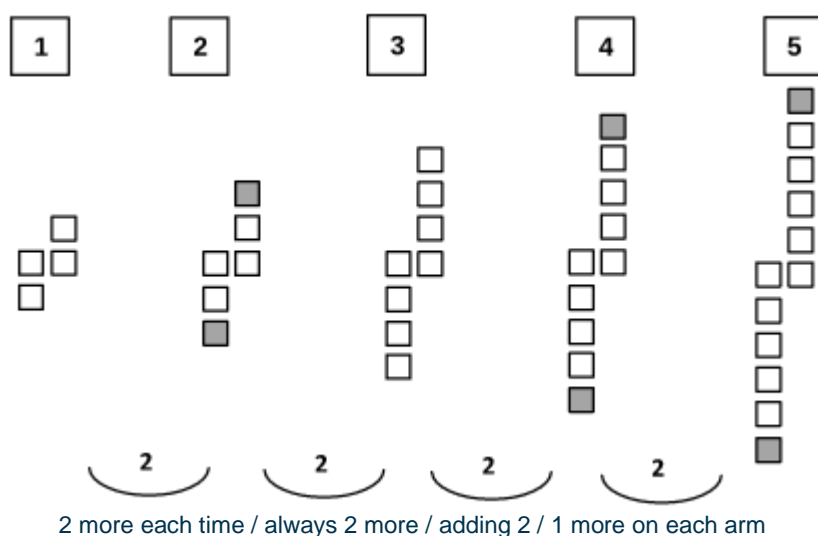
Finding the next figure in the pattern requires the child

- to recognise and apply the difference (growth) between consecutive figures or
- to find an explicit rule for each figure and use this explicitly for the looked for figure.

This 2nd worksheet in lesson 1 asks the children to find the next or a near figure. These activities often trigger recursive thinking, because the children only need to compare pairs of figures (the 2nd and the 3rd) to come up with an idea of change and the sequence of differences. This is appropriate at this stage, and should be discussed. It might be worthwhile to suggest to children that there will be other ways to explore the patterns in subsequent lessons, but that looking at the difference between them works for this one today.



3 more each time / always 3 more / an extra column (of 3) each time / adding 3



The initial approach to look for the differences between two consecutive figures focusses on the differences in quantity (difference sequence, 2,2,2... or 3,3,3 ... in the given pattern). The relationship can be explored in both directions: it is possible to describe the growth or the shrinking. Arrows between two neighbour figures are helpful to indicate the recursive relation.

This recursive approach is not stressed later in the unit, because it might lead to misconception and over generalisations of the difference sequence. However, recursive thinking is one growth point in functional thinking. Moreover, the reactions of the children give the teacher important information about the thinking processes. If the children can find the next figure recursively, the teacher can build on the experience by stressing other and new ideas (explicit approaches) in the next lessons.

Lesson 2

Two Worksheets “Near and far.”

Big Ideas

- Finding near figures in the pattern (specialising).
- Finding far figures in the pattern (generalising).

The position numbers (7th and 947th) of the figures asked for are chosen randomly. It is possible to ask for the 10th or 100th figure as well. The odd position numbers should encourage the children not to calculate quantities of squares or sticks by using multiples of a given figure, but to apply a rule they can identify in the pattern.

Near figures are not next to the given ones. Identifying near figures requires children to improve on their initial approaches from the first lesson. Recursive approaches may still work for the near figure (7th figure) but not for the far figure.

Possible answers:

In *figural* approaches the answers will refer to the spatial structure of the pattern:

- ‘one column of 2 and the number in the bottom row’ (pattern A)
- ‘one more than the number in a row and another one above’ (pattern A)
- ‘up and bottom the number and two more to close the figure’ (pattern B)
- ‘starting with a square, then every time one more in the top row and one more in the bottom row’ (pattern B, recursive)
- two horizontal line made of the number and two vertical lines’ (pattern B)

Some children may try to describe the situation in a *numerical* approach:

- ‘2 plus the number on the card’ (pattern A)
- ‘the number and 2 more’ (pattern A)
- ‘double the number and 2 more’ (pattern B)

Solutions, which use a figural approach, may refer to the whole pattern and not to a special figure. The children actually see structures and sub-structures into the pattern and often use spatial description, e.g. prepositions (up, above, next to, bottom, ...). Because this approach is close to a general rule, and in this sense more sophisticated than a solely numerical expression, they should be included when collecting ideas on the board at the end of the lesson.

The answers noted may include drawings of the figures, which is one possibility to solve the task for the near position number. The drawing approach is not helpful for the far figure however. Hence, the children should be encouraged to come up with a different idea.

The different ideas are shared in classroom discussion. The teacher ensures that everyone is allowed to check the ideas of others in order to understand the different ideas.

Table 2: Possible collection of ideas on the board: worksheet 1

1	2	3	...	7	...	947
1+2	1+3	1+4		1+8		1+948
<i>"I took one more than the figure number, and added another 1"</i>						
2+1	2+2	2+3		2+7		2+947
<i>"one column of 2 in every figure, plus the figure number in the bottom row"</i>						
<i>2 + the figure number</i>						
	→+1	→+1		→+1 4 times		

Collect the ideas of the children on the board (visualiser, projector, whiteboard). The figural approaches (mostly expressed verbally) are powerful tools, which focus on the structure of a pattern, and are valued in the plenary. As mentioned above, in numerical notations it is crucial not to note quantities of the figures but the expressions used (how would you calculate it?). For example, if a child starts to suggest 3, 4, 5, ask how did you get 3, 4, 5, and capture their ideas while keeping the structure evident:

"I added two to the figure number", teacher writes: 2+1, 2+2, 2+3

"I took one more than the figure number, and added another 1", teacher writes 2+1, 3+1, 4+1

Children in primary school will not be fluent in expressing their ideas using abstract symbols, such as letter variables, x, y, etc. Hence, the children need to come up with individual ideas, to exemplify their thinking. Research findings (Steinweg et al., 2018) proved various possibilities: If the children use position numbers in their descriptions, these can be meant as quasi-variables. Children regard the position number 947 not as a specific number but typify thereby all numbers or at least all numbers to come. Sometimes verbal indicators like always, every time and so forth signify quasi-variables. In other cases, listing several examples or conditional sentences (if there is seven, I add two to the seven) indicate developments in functional thinking.

Lesson 3

Two worksheets “Colour me!”

Big Ideas

- Identifying the two important elements (constant and change) of an explicit expression by
 - (a) colouring the constant element
 - (b) identifying the position number in each figure.

A linear function can be described in symbolic expressions, such as $m \times n + t$. The t is the constant element and m indicates the rate of change (slope), i.e., how many times n is used. To find a general rule for a pattern includes identifying both elements (constant and change).

From the child’s perspective the description, therefore, includes how many multiples of the position number can be seen in each figure, and a possible constant element.

The informal generalisation worked on in the previous lessons is taken up and improved. This lesson stresses the two important elements in the generalising process. Marking by colour or by circles the elements in the given pattern supports figural approaches, which are more effective than solely numerical ones.

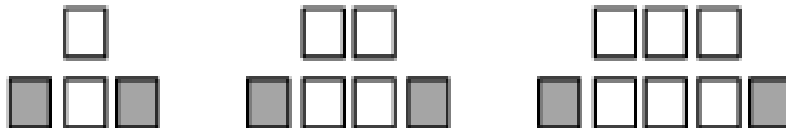
The patterns provided can be structured differently. Again, the individual working time is essential to allow children to come up with different ideas.

When a constant is identified, it must be identifiable in each figure of the pattern (see examples below). It might be easiest to describe the constant as a sub-figure within each pattern figure. Within each figure of the pattern this constant sub-figure must retain its shape, as well as the number of elements.

The identification of constant elements affects the relationship between the unshaded squares and the figure number. The descriptions given in task b need to suit the figural pattern. The descriptions can use verbal expressions or notations in numbers. The quantity of squares (shaded or not) is not relevant, but the appropriate description is. The individual descriptions do not have to be formal. The teacher uses the formal terms ‘position number’ and ‘constant’ in the discussion, however, in order to enable the children to steadily absorb this mathematical language.

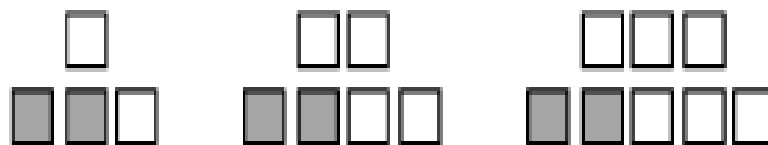
As already introduced in the previous lessons, notations displayed on the board (visualiser, projector, whiteboard) may support the exchange of different ideas and facilitate flexibility in recognising pattern elements.

Examples for colouring the constant element and describing the relationship with the position number (worksheet 1):



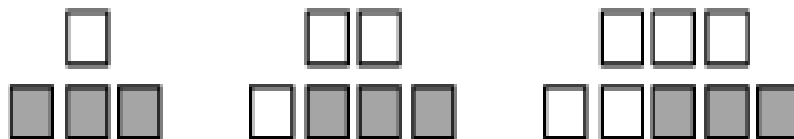
Constant element 1

“two left and right and double the number in the middle” leading to the rule $2 \times n + 2$



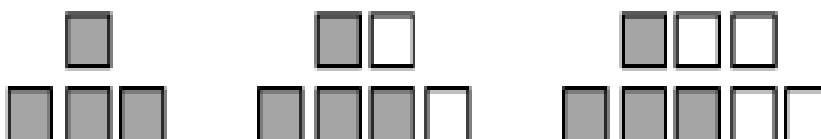
Constant element 2

“bottom row 2 (constant) and one times the (position) number, upper row the number” leading to the rule $2 + n + n$ leading to $2 + 2 \times n$



Constant element 3

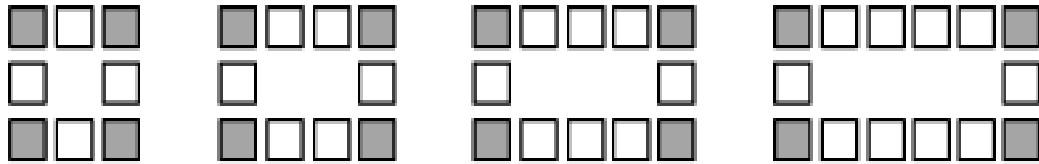
“Bottom row 3 and one less than the number, upper row the number” leading to the rule $n + (n - 1) + 3$ leading to $2 \times n - 1 + 3: 2 \times n + 2$



Constant element 4

“the first figure stays in all figures, and two time one less than the number is added to every row” leading to the rule $4 + 2 \times (n - 1)$

Examples for colouring the constant element and describing the relation to the position number (worksheet 2):



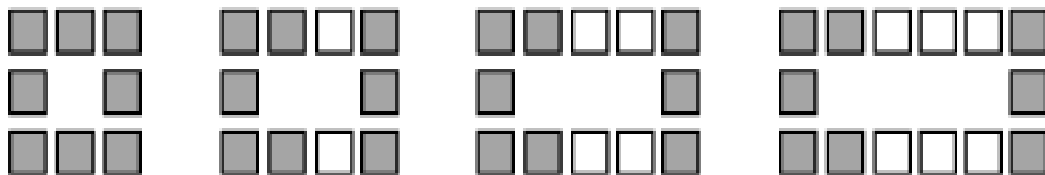
Constant element 1

“one in each corner, one more left and right (vertical) and two times the number in the bottom and top (horizontal) row” leading to the rule $4 + 2 + 2 \times n$



Constant element 2

“3 left and right (vertical) and two times the number in the bottom and top (horizontal) row” leading to the rule $6 + 2 \times n$



Constant element 3

“The first figure (8 squares) is enlarged in the bottom and top (horizontal) row by one less than the number” leading to the rule $8 + 2 \times (n-1)$

Lesson 4

Two Worksheets “Rules rule!”

Big Ideas:

- Finding an explicit rule.
- Justifying the rule, e.g., by colouring.
- Comparing patterns (with the same rate of change) and their rules.

The elements of a function rule are addressed in the previous lesson. The tasks in lesson 3 scaffold individual descriptions by separately focussing on the two elements of a formal notation (constant and rate of change /relationship to the position number). Now the focus is on the rule itself.

(1) The 1st worksheet in this lesson expects the children to make use of these earlier experiences (constant and relationship to the position number). The avatar (Jonas) sets an example. Again, different rules may suit the pattern.

(2) The identified rule then needs to be justified. The prompt to show the rule tries to entice the children to use figural approaches of justification and to capitalize on the drawing (geometrical pattern) given. The validation by colouring is obvious. However, verbal descriptions are also possible. The teacher makes sure that the descriptions suit the figural pattern and its structure and expects the children to show the described structures in the pattern.

Possible answers (worksheet 1)



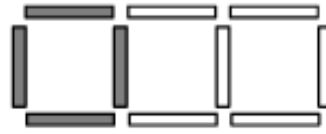
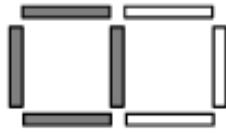
One vertical stick and three per (position) number

$$1 + 3 \times n$$

or

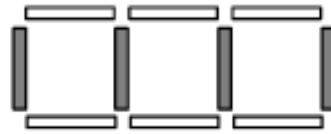
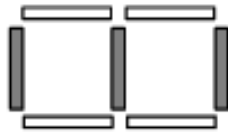
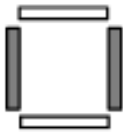
each square has 4, but the (vertical) doubles in the middle are not needed

$$1 + n \times (4-1)$$



The first figure remains constant and 3 times one less than the (position) number more

$$4 + (n-1) \times 3 \quad \text{or} \quad 4 + (n-1) \times (4-1)$$



Vertically one more than the (position) number and horizontally double the (position) number $(n+1) + 2 \times n$

As already introduced in the previous lessons, notations on the board and the explicit use of mathematical terms support the exchange of different ideas and facilitate flexibility in recognising pattern elements.

The 2nd worksheet asked for the rules of two patterns A and B with different figural appearance. However, the mathematical rule links the patterns, as both have the same rate of change.



$$2 \times n + 1$$



$$2 \times n + 4$$

There is no need to introduce the mathematical term *slope*, but rather to emphasise the power of figural approaches in identifying the structure.

The children can recognize and indicate this relationship by colouring or verbal descriptions. The differences (constant) and the similarities can be either shadowed or circled or both.



Constant elements shadowed and arms (of the angle) identified as double the position number

$$2 \times n + 1$$

$$2 \times n + 4$$



double the position number ($2 \times n$) circled in the 4th figure of both patterns

The pattern B used in this last worksheet of the unit, was the first pattern on worksheet one of the first lesson. In lesson #1 the pattern had introduced position numbers indicated by number cards, now it is worked on in depth. Hence, the pattern frames the unit. Working on this pattern again paradigmatically summarizes the development of functional thinking:

Recognising geometrical patterns as sequences of ordered growing figures, for which next, near, and far figures can be found, identifying of constant and change, and last, but not least, verbally or formally giving an explicit rule.




3

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The background features a collage of abstract geometric shapes. On the left, there are concentric green and light green arches. On the right, there are green leaf-like shapes and a green vertical stem against a light orange background. A large dark blue rectangle occupies the bottom right portion of the page, containing the main title.

4

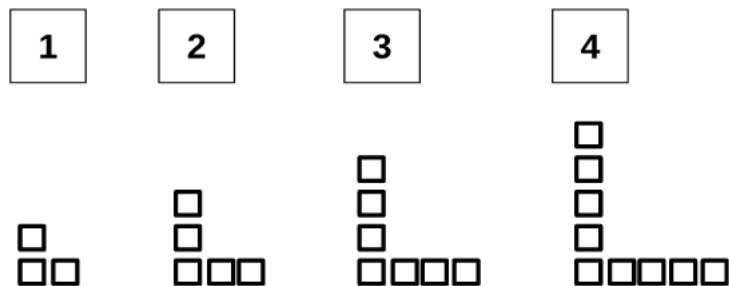
Appendix: Worksheets for Lessons 1-4

Appendix: Printable Worksheets for Lessons 1 - 4

Lesson 1, Part 1: Each figure has a number.

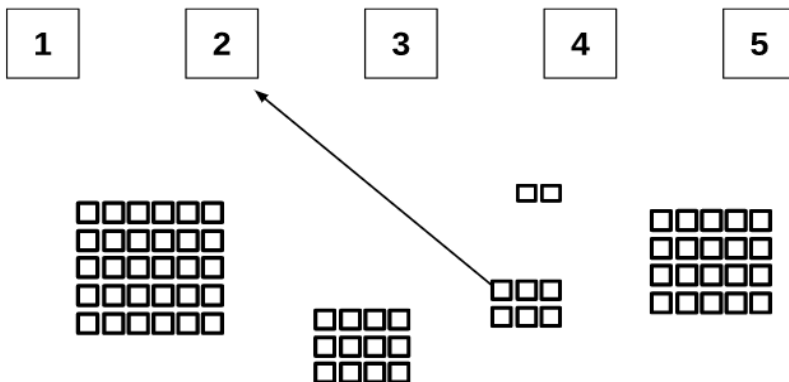
Name _____

A How does the position number (the number on the card) match to each figure?



Explain your thinking:

B All mixed up. Find the fitting position number.



Explain your thinking:

Lesson 1, Part 2: Next please!

Name _____

A Find the 4th figure.

1



2



3



4

Explain your thinking:

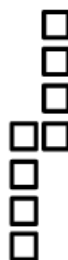
B Find the missing figures.

1



2

3



4

5

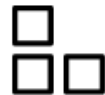


Explain your thinking:

Lesson 2: Near and Far, Part 1

Name _____

1



2



3



Describe the 7th figure.

Explain your thinking:

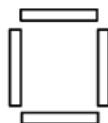
Describe the 947th figure

Explain your thinking:

Lesson 2, Near and Far: Part 2

Name _____

1



2



3



Describe the 7th figure.

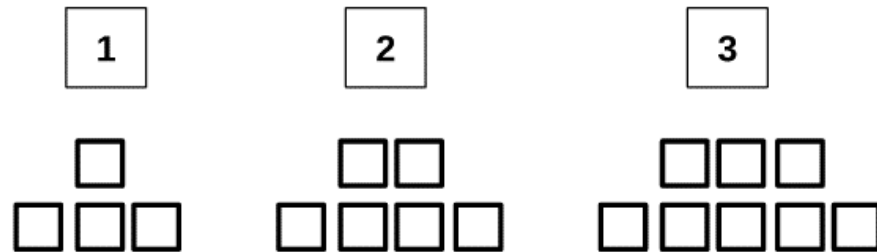
Explain your thinking:

Describe the 947th figure.

Explain your thinking:

Lesson 3, Colour Me! Part 1

Name _____



a) What stays the same? Colour the constant squares. Draw the 4th figure and colour the constant squares.

Explain your thinking:

b) Can you see a link to the position number in the squares that have **not** been coloured?

Explain your thinking:

Lesson 3, Colour Me! Part 2

Name _____

1



2



3



4



5

a) What stays the same? Colour the constant squares. Draw the 5th figure and colour the constant squares.

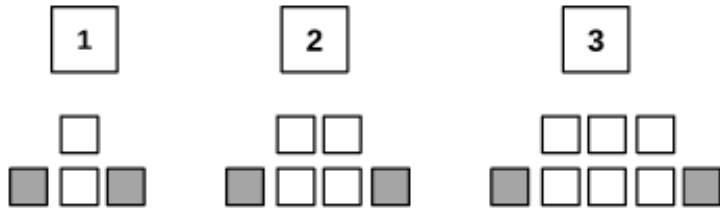
Explain your thinking:

b) Can you see a link to the position number in the squares that have **not** been coloured?

Explain your thinking:

Lesson 4, Rules rule! Part 1

Name _____

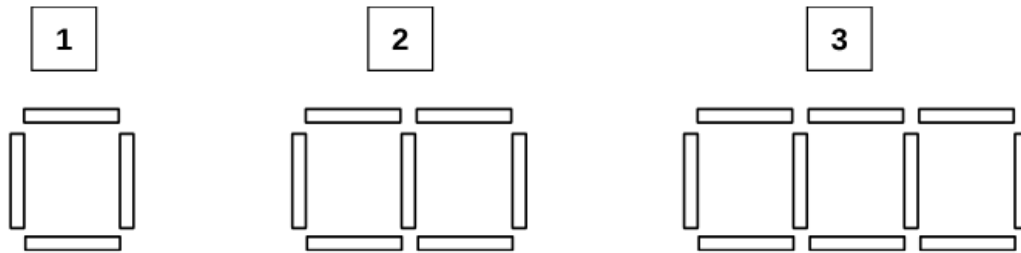


Jonas

My rule for this pattern is:

*Double the number
and two more.*

Your turn: Can you think of a rule for this pattern? Show me how your rule works.



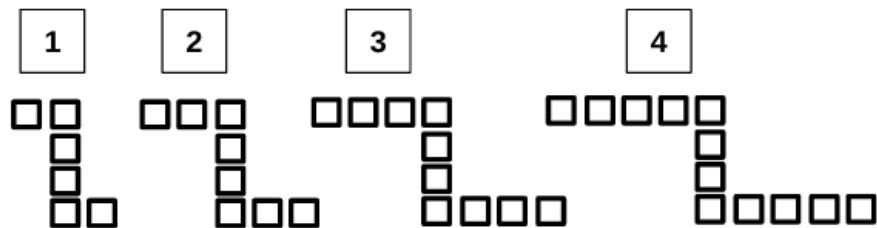
Explain your thinking:

Lesson 4, Rules rule! Part 2

Name _____

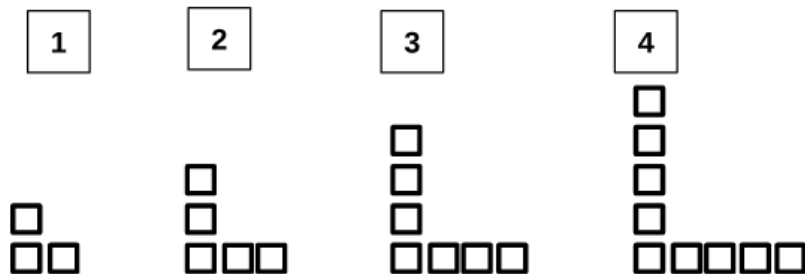
Can you think of a rule for each of these patterns?

A



Explain your thinking:

B



Explain your thinking:

Compare your findings for patterns A and B. Does anything strike you?

